# 2021 AIME I Problem & Solution

# 2021 AIME I Problem 1

Zou and Chou are practicing their 100-meter sprints by running races against each other. Zou wins the first race, and after that, the probability that one of them wins a race is if they won the previous race but only if they lost the previous race. The probability that Zou will win exactly of the races is , where and are relatively prime positive integers. Find

小邹和小周两人进行了场米短跑比赛．小邹在第一场比赛中获胜, 此后的每场比赛, 在前一场比赛中获胜的人获胜的概率为, 但在前一场比赛中失利的人获胜概率只有．小邹在场比赛中恰好赢了场的概率是, 其中和是互质的正整数．求．

Solution 1 (Casework)

For the next five races, Zou wins four and loses one. There are five possible outcome sequences, and we will proceed by casework:

Case (1): Zou does not lose the last race.

The probability that Zou loses a race is and the probability that he wins the following race is For each of the three other races, the probability that he wins is

There are four such outcome sequences. The probability of one such sequence is

Case (2): Zou loses the last race.

The probability that Zou loses a race is For each of the four other races, the probability that he wins is

There is one such outcome sequence. The probability is

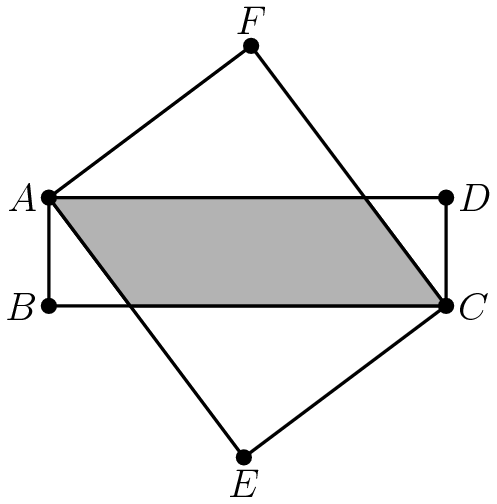
Answer

The requested probability is and the answer is

## 2021 AIME I Problem 2

In the diagram below, is a rectangle with side lengths and , and is a rectangle with side lengths and as shown. The area of the shaded region common to the interiors of both rectangles is , where and are relatively prime positive integers. Find .

在下图中，是边长为和的矩形，是边长为和的矩形．两个矩形内部的公共区域如阴影部分所示，面积为 其中和为互质的正整数．求．



Solution 1 (Similar Triangles) Let be the intersection of and . From vertical angles, we know that . Also, given that and are rectangles, we know that . Therefore, by AA similarity, we know that triangles and are similar.

Let . Then, we have . By similar triangles, we know that and . We have .

Solving for , we have . The area of the shaded region is just .

Thus, the answer is .

## 2021 AIME I Problem 3

Find the number of positive integers less than that can be expressed as the difference of two integral powers of

求可以表示成为两个的整数方幂之差的小于的正整数的个数.

Solution 1

We want to find the number of positive integers which can be written in the form for some non-negative integers (note that if , then ). We first observe must be at most 10; if , then . As , we can first choose two different numbers from the set in ways. This includes , , , , which are invalid as in this case. For all other choices and , the value of is less than 1000.

We claim that for all other choices of and , the values of are pairwise distinct. More specifically, if where and , we must show that . Suppose otherwise for sake of contradiction; rearranging yields . We use the fact that every positive integer has a unique binary representation:

If then ; from here we can deduce either and (contradicting the assumption that , or and (contradicting the assumption and ).

If then , and it follows that , also contradicting the assumption . Hence we obtain contradiction.\*

Then there are choices for for which is a positive integer less than 1000; by the above claim, each choice of results in a different positive integer . Then there are integers which can be expressed as a difference of two powers of 2.

Note: The uniqueness of binary representation could be rather easily proven, but if you cannot convince yourself on the spot that this is the case, consider the following alternative proof. Let where and and , for the sake of contradiction. Therefore , or . Plugging in, we see that , or , contradiction. Note by Ross Gao

Solution 2 (Casework)

Case 1: When our answer is in the form , where is an integer such that .

We start with the subcase where it is , for some integer where (this is because the case where yields , which doesn’t work because it must be a positive integer.) Note that , and . Our answer needs to be less than , so the maximum possible result (in this case) is . Our lowest result is . All the positive powers of two less than work, so we have possibilities for this subcase. For subcases and , we have and possibilities, respectively.

Case 2: When our answer is in the form of , where is an integer such that .

We can start with the subcase where . We notice that , and which is less than , so the greatest result in this subcase is actually , and the lowest is . Thus, we have possibilities. For the other four subcases, we have and possibilities, respectively.

Answer: We note that these are our only cases, as numbers in the form of and beyond are greater than .

Thus, our result is . ~jehu26

Solution 3

First, you need to notice that it is impossible to have overlapping, making the problem easier.

Case 1 : There are ways here, from to

It is easy to see here that this continues all the way down to one. However, when the case gets to , there are 5 ways instead of 4 because is smaller than 1000.

Thus, So the answer is

## 2021 AIME I Problem 4

Find the number of ways identical coins can be separated into three nonempty piles so that there are fewer coins in the first pile than in the second pile and fewer coins in the second pile than in the third pile.

将枚相同的硬币分成三堆, 每堆都有硬币, 并且第一堆的硬币比第二堆的少, 第二堆的硬币比第三堆的少. 问这样的分法共有多少种?

Solution 1

Suppose we have coin in the first pile. Then all work for a total of piles. Suppose we have coins in the first pile, then all work, for a total of . Continuing this pattern until coins in the first pile, we have the sum .

(Minor edit to make everything fit in the page made by KingRavi)

Solution 2

Let the three piles have coins respectively. If we disregard order, then we just need to divide by at the end.

We know . Since are positive integers, there are ways from Stars and Bars.

However, we must discard the cases where or or . The three cases are symmetric, so we just take the first case and multiply by 3. We have for 32 solutions. Multiplying by 3, we will subtract 96 from our total.

But we undercounted where . This is first counted 1 time, then we subtract it 3 times, so we add it back twice. There is clearly only 1 way, for a total of 2.

Hence, the answer is

Solution 3

Let the piles have and coins, with . Then, let , and , such that each . The sum is then . This is simply the number of positive solutions to the equation . Now, we take cases on .

If , then . Each value of corresponds to a unique value of , so there are solutions in this case. Similarly, if , then , for a total of solutions in this case. If , then , for a total of solutions. In general, the number of solutions is just all the numbers that aren’t a multiple of , that are less than or equal to .

We then add our cases to getas our answer.

## 2021 AIME I Problem 5

Call a three-term strictly increasing arithmetic sequence of integers special if the sum of the squares of the three terms equals the product of the middle term and the square of the common difference. Find the sum of the third terms of all special sequences.

对于一个由三个整数组成的严格递增的等差数列, 如果三项的平方和等于中间项与公差平方的乘积, 那么它称为特殊的. 求所有特殊数列的第三项之和.

Solution 1

Let the terms be , , and . Then we want , or . Rearranging, we get . Simplifying further, . Looking at this second equation, since the right side must be an integer, must equal . Looking at the first equation, we see since is positive. This means we must test . After testing these, we see that only and work which give and respectively. Thus the answer is . ~JHawk0224

Solution 2

Let the common difference be and let the middle term be . Then, we have that the sequence isThis means that the sum of the squares of the 3 terms of the sequence isWe know that this must be equal to so we can write thatand it follows that Now, we can treat as a constant and use the quadratic formula to getWe can factor pull out of the square root to getHere, it is easy to test values of . We find that and are the only positive integer values of that make a positive integer. (To prove this, let , then which is then remembering that and are integers see if you can figure it out. -PureSwag) gives and , but we can ignore the latter. gives , as well as a fraction which we can ignore.

Since and are the only two solutions and we want the sum of the third terms, our answer is . -BorealBear, minor edit by Kinglogic

Solution 3

Proceed as in solution 2, until we reach Write

, it follows that for some (positive) integer k and .

Taking both sides modulo , , so .

When , we have and . When , we have and .

Summing the two cases, we have . ~Ross Gao

Solution 4 (Combining Solution 1 and Solution 3)

As in Solution 1, write the three integers in the sequence as , , and .

Then the sum of the squares of the three integers is .

Setting this equal to the middle term times the common difference squared, which is ,

and solving for we get:

The numerator has to be positive, so the denominator has to be positive too for the sequence

to be strictly increasing; that is, .

For to be a perfect square, must be a perfect square as well.

This means that is divisible by 3, and whatever left over is a perfect square.

We can express this as an equation: let the perfect square left over be . Then:

. Now when you divide the numerator and denominator by 3, you are left with

. Because the sequence is of integers, d must also be an

integer, which means that must divide .

Taking the above equation we can solve for : .

This means that is divisible by . is automatically divisible by , so

must be divisible by . Then must be either of . Plugging back into the equation,

, so .

, so .

Finally,

## 2021 AIME I Problem 6

Segments and are edges of a cube and is a diagonal through the center of the cube. Point satisfies and . What is ?

线段和是一个立方体的边, 而线段是通过立方体中心的对角线. 点满足, 并且. 求的长度.

Solution 1

First scale down the whole cube by 12. Let point P have coordinates , A have coordinates , and be the side length. Then we have the equations. These simplify into .

Adding the first three equations together, we get . Subtracting this from the fourth equation, we get , so . This means . However, we scaled down everything by 12 so our answer is . ~JHawk0224

Solution 2 (Solution 1 with slight simplification)

Once the equations for the distance between point P and the vertices of the cube have been written. We can add the first, second, and third to receive, Subtracting the fourth equation gives, Since point , and since we scaled the answer is ~Aaryabhatta1

Solution 3

Let E be the vertex of the cube such that ABED is a square. By the British Flag Theorem, we can easily we can show thatandHence, adding the two equations together, we get . Substituting in the values we know, we get .

Thus, we can solve for , which ends up being .

## 2021 AIME I Problem 7

Find the number of pairs of positive integers with such that there exists a real number satisfying

求具有如下性质的正整数对的数目: , 并且存在实数满足

Solution 1

The maximum value of is , which is achieved at for some integer . This is left as an exercise to the reader.

This implies that , and that and , for integers .

Taking their ratio, we haveIt remains to find all that satisfy this equation.

If , then . This corresponds to choosing two elements from the set . There are ways to do so.

If , by multiplying and by the same constant , we have that . Then either , or . But the first case was already counted, so we don’t need to consider that case. The other case corresponds to choosing two numbers from the set . There are ways here.

Finally, if , note that must be an integer. This means that belong to the set , or . Taking casework on , we get the sets . Some sets have been omitted; this is because they were counted in the other cases already. This sums to .

In total, there are pairs of .

This solution was brought to you by Leonard\_my\_dude

Solution 2

In order for , .

This happens when mod

This means that and for any integers and .

As in Solution 1, take the ratio of the two equations: Now notice that the numerator and denominator of are both odd, which means that and have the same power of two (the powers of 2 cancel out).

Let the common power be : then , and where and are integers between 1 and 30.

We can now rewrite the equation: Now it is easy to tell that mod and mod . However, there is another case: that

mod and mod . This is because multiplying both and by will not change the fraction, but each congruence will be changed to mod mod .

From the first set of congruences, we find that and can be two of .

From the second set of congruences, we find that and can be two of .

Now all we have to do is multiply by to get back to and . Let’s organize the solutions in order of increasing values of , keeping in mind that and are bounded between 1 and 30.

For we get .

For we get

For we get

If we increase the value of more, there will be less than two integers in our sets, so we are done there.

There are 8 numbers in the first set, 7 in the second, 4 in the third, 4 in the fourth, 2 in the fifth, and 2 in the sixth.

In each of these sets we can choose 2 numbers to be and and then assign them in increasing order. Thus there are:

possible pairings of that satisfy the conditions.

-KingRavi

Solution 3

We know that the range of is between and .

Thus, the only way for the sum to be is for of and to both be .

The of is equal to 1.

Assuming and are both positive, m and n could be . There are ways, so .

If bother are negative, m and n could be . There are ways, so .

However, the pair could also be and so on. The same goes for some other pairs.

In total there are of these extra pairs.

The answer is

## 2021 AIME I Problem 8

Find the number of integers such that the equationhas distinct real solutions.

求使得方程有个不同的实数解的整数的个数.

Solution 1

Let Then the equation becomes , or . Note that since , is nonnegative, so we only care about nonnegative solutions in . Notice that each positive solution in gives two solutions in (), whereas if is a solution, this only gives one solution in , . Since the total number of solutions in is even, must not be a solution. Hence, we require that has exactly positive solutions and is not solved by

If , then is negative, and therefore cannot be the absolute value of . This means the equation’s only solutions are in . There is no way for this equation to have solutions, since the quadratic can only take on each of the two values at most twice, yielding at most solutions. Hence, . also can’t equal , since this would mean would solve the equation. Hence,

At this point, the equation will always have exactly positive solutions, since takes on each positive value exactly once when is restricted to positive values (graph it to see this), and are both positive. Therefore, we just need to have the remaining solutions exactly. This means the horizontal lines at each intersect the parabola in two places. This occurs when the two lines are above the parabola’s vertex . Hence we have: Hence, the integers satisfying the conditions are those satisfying There are such integers. Note: Be careful of counting at the end, you may mess up and get 59.

Solution 2 (also graphing)

Graph (If you are having trouble, look at the description in the next two lines and/or the diagram in solution 3). Notice that we want this to be equal to and .

We see that from left to right, the graph first dips from very positive to at , then rebounds up to at , then falls back down to at .

The positive are symmetric, so the graph re-ascends to at , falls back to at , and rises to arbitrarily large values afterwards.

Now we analyze the (varied by ) values. At , we will have no solutions, as the line will have no intersections with our graph.

At , we will have exactly solutions for the three zeroes.

At for any strictly between and , we will have exactly solutions.

At , we will have solutions, because local maxima are reached at .

At , we will have exactly solutions.

To get distinct solutions for , both and must produce solutions.

Thus and , so is required.

It is easy to verify that all of these choices of produce distinct solutions (none overlap), so our answer is .

Solution 3 (Piecewise Functions: Analyses and Graphs)

We take cases for the outermost absolute value, then rearrange:

Let We will rewrite as a piecewise function without using any absolute value:

We graph as shown below, with some key points labeled. The fact that is an even function ( holds for all real numbers from which the graph of is symmetric about the -axis) should facilitate the process of graphing.

Since has distinct real solutions, it is clear that each case has distinct real solutions geometrically. We shift the graph of down by units, where

For to have distinct real solutions, we need

For to have distinct real solutions, we need

Taking the intersection of these two cases gives from which there are such integers ~MRENTHUSIASM

Solution 4

Removing the absolute value bars from the equation successively, we get

The discriminant of this equation is

Equating the discriminant to , we see that there will be two distinct solutions to each of the possible quadratics above only in the interval . However, the number of zeros the equation has is determined by where and intersect, namely at . When , , will have only solutions, and when , , then there will be real solutions, if they exist at all. In order to have solutions here, we thus need to ensure , so that exactly out of the possible equations of the form given above have y-intercepts below and only real solutions, while the remaining equations have solutions. This occurs when , so our final bounds are , giving us valid values of .

## 2021 AIME I Problem 9

Let be an isosceles trapezoid with and Suppose that the distances from to the lines and are and respectively. Let be the area of Find

设为等腰梯形, , 并且. 假设从到直线和的距离分别为和. 设为的面积, 求的值.

Solution 1

Construct your isosceles trapezoid. Let, for simplicity, , , and . Extend the sides and mark the intersection as . Following what the question states, drop a perpendicular from to labeling the foot as . Drop another perpendicular from to , calling the foot . Lastly, drop a perpendicular from to , labeling it . In addition, drop a perpendicular from to calling its foot .

–DIAGRAM COMING SOON–

Start out by constructing a triangle congruent to with its side of length on line . This works because all isosceles triangles are cyclic and as a result, .

Notice that by AA similarity. We are given that and by symmetry we can deduce that . As a result, . This gives us that .

The question asks us along the lines of finding the area, , of the trapezoid . We look at the area of and notice that it can be represented as . Substituting , we solve for , getting .

Now let us focus on isosceles triangle , where . Since, is an altitude from to of an isosceles triangle, must be equal to . Since and , we can solve to get that and .

We must then set up equations using the Pythagorean Theorem, writing everything in terms of , , and . Looking at right triangle we getLooking at right triangle we getNow rearranging and solving, we get two equationThose are convenient equations as which gives usAfter some “smart” calculation, we get that .

Notice that the question asks for , and by applying the trapezoid area formula. Fortunately, this is just , and plugging in the value of , we get that . ~Math\_Genius\_164

Solution 2 (LOC and Trig)

Call AD and BC . Draw diagonal AC and call the foot of the perpendicular from B to AC . Call the foot of the perpendicular from A to line BC F, and call the foot of the perpindicular from A to DC H. Triangles CBG and CAF are similar, and we get that Therefore, . It then follows that triangles ABF and ADH are similar. Using similar triangles, we can then find that . Using the Law of Cosine on ABC, We can find that the cosine of angle ABC is . Since angles ABF and ADH are equivalent and supplementary to angle ABC, we know that the cosine of angle ADH is 1/3. It then follows that . Then it can be found that the area is . Multiplying this by , the answer is . -happykeeper

Solution 3 (Similarity)

Let the foot of the altitude from A to BC be P, to CD be Q, and to BD be R.

Note that all isosceles trapezoids are cyclic quadrilaterals; thus, is on the circumcircle of and we have that is the Simson Line from . As , we have that , with the last equality coming from cyclic quadrilateral . Thus, and we have that or that , which we can see gives us that . Further ratios using the same similar triangles gives that and .

We also see that quadrilaterals and are both cyclic, with diameters of the circumcircles being and respectively. The intersection of the circumcircles are the points and , and we know and are both line segments passing through an intersection of the two circles with one endpoint on each circle. By Fact 5, we know then that there exists a spiral similarity with center A taking to . Because we know a lot about but very little about and we would like to know more, we wish to find the ratio of similitude between the two triangles.

To do this, we use the one number we have for : we know that the altitude from to has length 10. As the two triangles are similar, if we can find the height from to , we can take the ratio of the two heights as the ratio of similitude. To do this, we once again note that . Using this, we can drop the altitude from to and let it intersect at . Then, let and thus . We then have by the Pythagorean Theorem on and : Then, . This gives us then from right triangle that and thus the ratio of to is . From this, we see then thatandThe Pythagorean Theorem on then gives that Then, we have the height of trapezoid is , the top base is , and the bottom base is . From the equation of a trapezoid, , so the answer is . ~lvmath

Solution 4 (Cool Solution by advanture)

First, draw the diagram. Then, notice that since is isosceles, , and the length of the altitude from to is also . Let the foot of this altitude be , and let the foot of the altitude from to be denoted as . Then, . So, . Now, notice that , where denotes the area of triangle . Letting , this equality becomes . Also, from , we have . Now, by the Pythagorean theorem on triangles and , we have and . Notice that , so . Squaring both sides of the equation once, moving and to the right, dividing both sides by , and squaring the equation once more, we are left with . Dividing both sides by (since we know is positive), we are left with . Solving for gives us .

Now, let the foot of the perpendicular from to be . Then let . Let the foot of the perpendicular from to be . Then, is also equal to . Notice that is a rectangle, so . Now, we have . By the Pythagorean theorem applied to , we have . We know that , so we can plug this into this equation. Solving for , we get .

Finally, to find , we use the formula for the area of a trapezoid: . The problem asks us for , which comes out to be . ~advanture

Solution 5 (Compact similarity solution)

Let and be the feet of the altitudes from to and , respectively.

Claim: We have pairs of similar right triangles: and .

Proof: Note that is cyclic. We need one more angle, and we get this from this cyclic quad: Let . We obtain from the similarities and .

By Ptolemy, , so .

We obtain , so .

Applying the Pythagorean theorem on , we get .

Thus, , and , yielding .

Solution 6 (Similar Triangles, Two Variables, Two Equations)

Let and be the perpendiculars from to and respectively. Next, let be the intersection of and

We set and as shown below.

From here, we obtain by segment subtraction, and and by the Pythagorean Theorem.

Since and are both complementary to we have from which by AA. It follows that or

Since by vertical angles, we have by AA, with ratio of similitude It follows that

Since by angle chasing, we have by AA, with ratio of similitude It follows that

By the Pythagorean Theorem on right we have or

Solving this system of equations ( and ), we get and from which and Finally, the area of isfrom which

## 2021 AIME I Problem 10

Consider the sequence of positive rational numbers defined by and for , if for relatively prime positive integers and , then Determine the sum of all positive integers such that the rational number can be written in the form for some positive integer .

由有理数组成的数列按如下方式定义: , 并且对于, 如果 对于互质的正整数和成立, 那么 . 求能够找到正整数, 使得有理数i可以写成 形式的所有正整数的总和.

Solution

We know that when so is a possible value of .

Note also that for .

Then unless and are not relatively prime which happens when divides or divides , so the least value of is and .

We know . Now unless and are not relatively prime which happens the first time divides or divides or , and .

We have . Now unless and are not relatively prime.

This happens the first time divides implying divides , which is prime so and . We have .

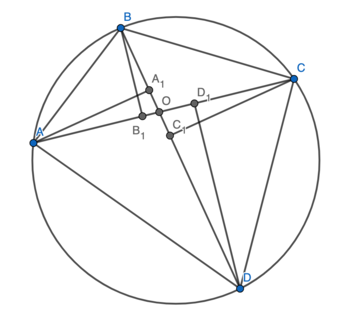
We have , which is always reduced by EA, so the sum of all is .

## 2021 AIME I Problem 11

Let be a cyclic quadrilateral with and . Let and be the feet of the perpendiculars from and , respectively, to line and let and be the feet of the perpendiculars from and respectively, to line . The perimeter of is , where and are relatively prime positive integers. Find .

设为圆内接四边形, , 并且. 设和分别是和到直线的垂线的垂足, 设和分别是和到直线的垂线的垂足. 周长为, 其中和为互质的正整数, 求.

Solution



Let be the intersection of and . Let .

Firstly, since , we deduce that is cyclic. This implies that , with a ratio of . This means that . Similarly, . HenceIt therefore only remains to find .

From Ptolemy’s theorem, we have that . From Brahmagupta’s Formula, . But the area is also , so . Then the desired fraction is for an answer of .

Finding 2

The angle between diagonals satisfies (see https://en.wikipedia.org/wiki/Cyclic\_quadrilateral#Angle\_formulas).

Thus,or. That is, or Thus, or In this context, .

Thus, ~y.grace.yu

Solution 3 (Pythagorean Theorem)

We assume that the two quadrilateral mentioned in the problem are similar (due to both of them being cyclic). Note that by Ptolemy’s, one of the diagonals has length WLOG we focus on diagonal To find the diagonal of the inner quadrilateral, we drop the altitude from and and calculate the length of Let be (Thus By Pythagorean theorem, we haveNow let be (thus making ). Similarly, we haveWe see that , the scaled down diagonal is just which is times our original diagonal implying a scale factor of Thus, due to perimeters scaling linearly, the perimeter of the new quadrilateral is simply making our answer -fidgetboss\_4000

## 2021 AIME I Problem 12

Let be a dodecagon (12-gon). Three frogs initially sit at and . At the end of each minute, simultaneously, each of the three frogs jumps to one of the two vertices adjacent to its current position, chosen randomly and independently with both choices being equally likely. All three frogs stop jumping as soon as two frogs arrive at the same vertex at the same time. The expected number of minutes until the frogs stop jumping is , where and are relatively prime positive integers. Find .

设是一个十二边形, 三只青蛙分别坐在和处. 每分钟结束时, 三只青蛙同时分别跳到与其当前位置相邻的两个顶点中的一个, 这两个顶点随机独立的选择, 可能性相同. 只要有两只青蛙同时到达同一个顶点, 那么所有三只青蛙就会停止跳跃. 从开始到青蛙停止跳跃所需的分钟数的期望为, 其中和为互质的正整数, 求.

Solution

The expected number of steps depends on the distance between the frogs, not on the order in which these distances appear. Let where denote the expected number of steps that it takes for two frogs to meet if traversing in clockwise or counterclockwise order, the frogs are , and vertices apart. Then

, giving ; (1)

, giving ; (2)

, giving ; (3)

Plug in (1) and (3) into (2), we see that . .

Each step is one minute. The answer is .

## 2021 AIME I Problem 13

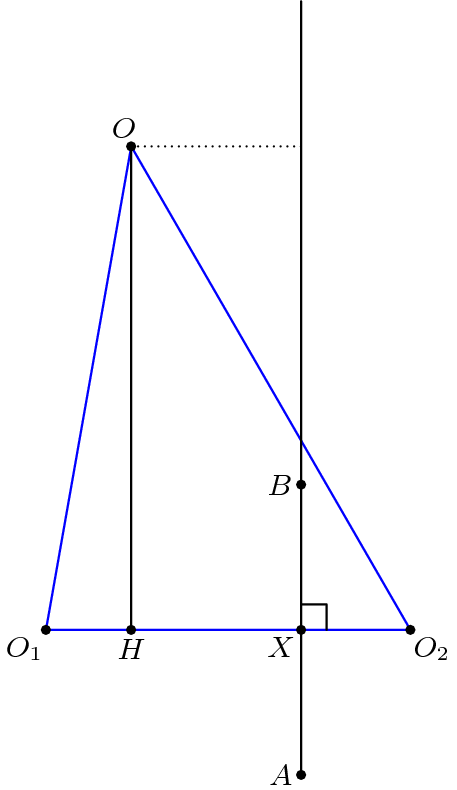
Circles and with radii and , respectively, intersect at distinct points and . A third circle is externally tangent to both and . Suppose line intersects at two points and such that the measure of minor arc is . Find the distance between the centers of and .

圆和的半径分别为和, 两圆交于不同的点和. 第三个圆与和相外切. 假设直线与相交于两点和, 使得劣弧是, 求$\_ { 1 }\_ { 2 }$的圆心之间的距离.

Solution

Let and be the center and radius of , and let and be the center and radius of .

Since extends to an arc with arc , the distance from to is . Let . Consider . The line is perpendicular to and passes through . Let be the foot from to ; so . We have by tangency and . Let .



Since is on the radical axis of and , it has equal power with respect to both circles, so

since .

Now we can solve for and , and in particular,

We want to solve for . By the Pythagorean Theorem (twice):

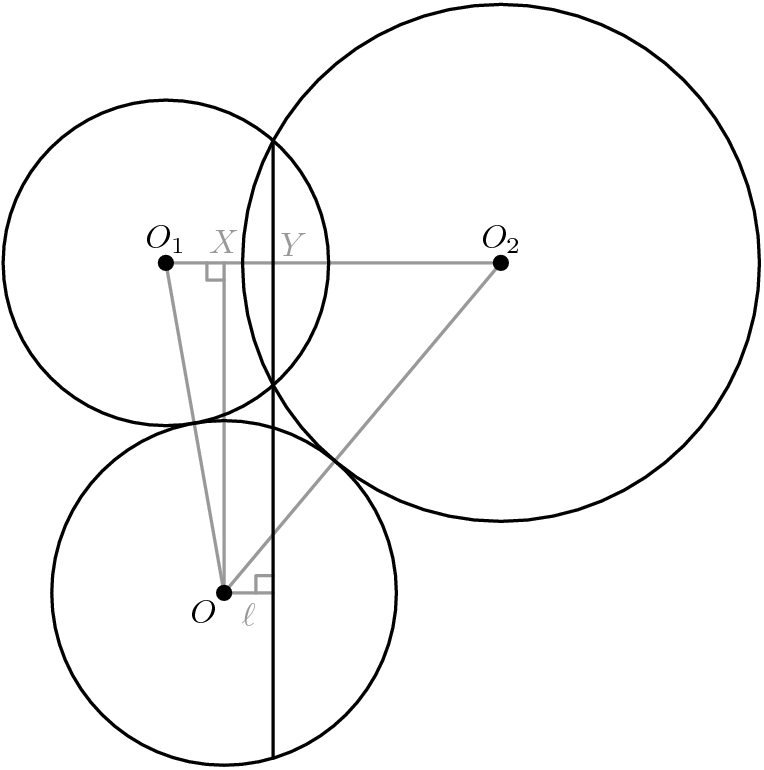
Therefore, .

Solution 2 (Official MAA, Unedited)

Denote by , , and the centers of , , and , respectively. Let and denote the radii of and respectively, be the radius of , and the distance from to the line . We claim that

where .

This solves the problem, for then the condition implies , and then we can solve to get .



Denote by and the centers of and respectively. Set as the projection of onto , and denote by the intersection of with . Note that . Now recall that

Furthermore, note that

Substituting the first equality into the second one and subtracting yields which rearranges to the desired.

## 2021 AIME I Problem 14

For any positive integer denotes the sum of the positive integer divisors of . Let be the least positive integer such that is divisible by for all positive integers . Find the sum of the prime factors in the prime factorization of .

对于任何正整数, 用表示的正整数因数之和. 令是最小的正整数, 使得对于所有的正整数可以被整除. 求的质因数分解中所有质因数的和.

Solution 1

We require that for all , so it is necessary and sufficient to ensure for all .

We solve the problem for primes:

Claim: A prime always divides if and only if divides .

Proof. For choices of , we need , i.e.  . This holds for all if and only if by Fermat’s little theorem.

If , we need . By lifting the exponent, this implies . It follows that we must have , since otherwise some choice of would force . We will verify is sufficient for to hold: indeed,

Therefore, the smallest is , and the requested sum is .

Solution 2

You get that (I’ll just do the case, but is the same)

. If , we have that . The minimum value of s.t. is where is the carmichael function.

When , we require that .

Symmetrically we also require that . So our answer is

## 2021 AIME I Problem 15

Let be the set of positive integers such that the two parabolas intersect in four distinct points, and these four points lie on a circle with radius at most . Find the sum of the least element of and the greatest element of .

使得两条抛物线和相交于四个不同点, 并且这四个点在某个半径不超过的圆上的正整数组成的集合记 为, 求中最小元素与中最大元素的和.

Solution

Make the translation to obtain . Multiply the first equation by 2 and sum, we see that . Completing the square gives us ; this explains why the two parabolas intersect at four points that lie on a circle\*. For the upper bound, observe that , so .

For the lower bound, we need to ensure there are 4 intersections to begin with. (Here I’m using the un-translated coordinates.) Draw up a graph, and realize that two intersections are guaranteed, on the so called “right branch” of . As we increase the value of k, two more intersections appear on the “left branch.”

does not work because the “leftmost” point of is which lies to the right of , which is on the graph . While technically speaking this doesn’t prove that there are no intersections (why?), drawing the graph should convince you that this is the case. Clearly, no k less than 4 works either.

does work because the two graphs intersect at , and by drawing the graph, you realize this is not a tangent point and there is in fact another intersection nearby, due to slope. Therefore, the answer is .

In general, (Assuming four intersections exist) when two conics intersect, if one conic can be written as and the other as for f,g polynomials of degree at most 1, whenever are linearly independent, we can combine the two equations and then complete the square to achieve . We can also combine these two equations to form a parabola, or a hyperbola, or an ellipse. When are not L.I., the intersection points instead lie on a line, which is a circle of radius infinity. When the two conics only have 3,2 or 1 intersection points, the statement that all these points lie on a circle is trivially true. ~Ross Gao